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On the Strong Secure Domination Number of a Graph

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Abstract: In this paper, we characterize trees with a strong secure domination number less than or equal to 4 and compute this parameter for certain classes of graphs. Also, we investigate bounds for the strong secure domination number of vertex gluing of two graphs.

Keywords: secure domination; strong secure domination; tree; vertex gluing

MSC: 05C69; 05C05

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1. Introduction

The graphs examined in this study are finite, simple, undirected, without loops, and with multiple edges. Let $G = (V, E)$ be a graph and let $v \in V(G)$. The number of vertices adjacent to v is called the *degree* of v and is denoted by $deg(v)$. The *open neighborhood* of v is defined as $N(v) = \{u : uv \in E(G)\}$. The *closed neighborhood* of v is defined as $N[v] = N(v) \cup \{v\}$. Let $X \subseteq V(G)$ and $u \in X$. The *private neighborhood* $pn(u, X)$ is defined by $pn(u, X) = \{v : N(v) \cap X = \{u\}\}$. The *external private neighborhood* of a vertex is defined as $epn(u, X) = \{v : v \in V \backslash X \text{ and } N(v) \cap X = \{u\}\}$, also represented by $epn_G(u, X)$. A connected acyclic graph is called a *tree*. A tree with a vertex of full degree is referred to as a *star* and is denoted by $K_{1,k}$, where k is the degree of the full-degree vertex. A vertex of degree one is known as a *pendant vertex*. A neighboring vertex to a pendant vertex is referred to as its *support vertex*. A *caterpillar* is a tree in which the removal of all its pendant vertices results in a path. The graph obtained by removing an edge e from a graph G is denoted by $G - e$ and this is called *edge deletion*. A path on n vertices and a cycle on n vertices are denoted P_n and C_n, respectively. The reader may refer to [1] for notation and terminology that is not defined here.

Let $D \subseteq V(G)$. If every vertex that is not in D is adjacent to a vertex that is in D, then D is a *dominating set*. The minimum cardinality of a dominating set in G is its *domination number* $\gamma(G)$ of G. A well-researched graph parameter in the literature is domination, we refer the reader to the book [2] and its references for more details.

Sampathkumar et al. [3] introduced the strong domination number in 1996. If for every $v \in V(G) \backslash D$ there is a vertex $u \in D$ with $uv \in E(G)$ and $deg(u) \geq deg(v)$, then a set D of vertices in a graph G is a *strong dominating set*. The minimum cardinality of a strong dominating set in G is its *strong domination number* $\gamma_{st}(G)$ of G. The results of a study on strong domination in graphs were presented in [4,5]. The concept of strong domination has, thus, been extensively studied in the literature. In [6], Cockayne et al. introduced the concept of secure domination. A dominating set D is called a *secure dominating set* (sds) if for every $v \in V \backslash D$ there exists some $u \in D$ such that u and v are adjacent and $N[(D \backslash \{u\}) \cup \{v\}] = V(G)$. Here, u is called a *defender* of v. The secure domination number $\gamma_{sd}(G)$ of G is the minimum cardinality of a secure dominating set in G. Additional findings regarding this parameter can be found in [7–10].

Inspired by the ideas of secure and strong domination, it is logical to explore both of these concepts further, beginning with the idea of strong secure domination as put forth by Annaal Mercy et al. [11]. A secure dominating set D is called a *strong secure dominating set* (ssds) if for every $v \in V \backslash D$ there exists $u \in D$ such that u and v are adjacent and $deg(u) \geq deg(v)$. The corresponding u is called a *strong neighbor* of v. The minimum cardinality of an ssds is called a *strong secure domination number* and is denoted by $\gamma_{ss}(G)$. Note that $\gamma_{ss}(G) = n$ if and only if G is a totally disconnected graph.

Why should such a domination in graphs be taken into consideration? This is the scenario: A computer network consists of a number of linked devices, each of which has connections to other devices. Consider a scenario where a network administrator must set up access control lists (ACLs) to manage network traffic flow. The following guidelines are what the administrator wishes to implement:

- Each device not explicitly allowed access to certain network resources should have at least one connected device that is granted access, ensuring connectivity and efficient data transmission.
- If a device is persistent about using a particular resource and applies pressure to be allowed access, it can be swapped out for a connected device that has already been given permission, as long as the new configuration keeps the connectivity requirement from the initial condition.
- It is recommended that any device that is not explicitly granted access to specific network resources be connected directly to a device that has higher access privileges. In this way, a device with greater permissions will always be accessible for potential administrative or troubleshooting needs. This implies that devices with limited access are purposefully connected to devices with more expansive access capabilities. This makes network management easier and guarantees that troubleshooting tasks can be completed successfully, which improves the overall security and resilience of the network.

The goal of the network administrator is to reduce the size of the access control lists while optimizing resource allocation and network security. This strategy is consistent with the strong secure dominating theory, which maintains network connectivity while attending to the needs of individual device access. This idea is useful in a variety of contexts, including the setting up of ATMs in multiple locations, the distribution of secret keys in cryptography, military operations, and commercial endeavors.

2. Preliminaries

First, let us display the strong secure domination number of standard graphs.

Remark 1. *For $n \in \mathbb{N}$, we have*

(i). $\gamma_{ss}(K_{1,n}) = n$.
(ii). $\gamma_{ss}(P_n) = \lceil \frac{3n}{7} \rceil$.
(iii). $\gamma_{ss}(C_n) = \lceil \frac{3n}{7} \rceil$.

The next two results are helpful in the sections that follow.

Proposition 1 ([12])**.** *Let G be a connected graph and X be a non-empty subset of V. Then, the following statements are equivalent.*

(i). *X is a secure dominating set.*
(ii). *For each $u \in V \backslash X$, there exists $v \in X \cap N(u)$ such that $epn(v, X) \subseteq N[u]$.*
(iii). *For each $u \in V \backslash X$, there exists $v \in X \cap N(u)$ such that the subgraph induced by $\{u, v\} \cup epn(v, X)$ is complete.*

The following proposition is obtained by using Proposition 1.

Proposition 2. *Let $G \ncong K_2$ be a graph. If a support vertex u in G has k pendant neighbors, then:*

(i). *Every sds must contain either u and $k-1$ of its pendant neighbors or k pendant neighbors.*
(ii). *Every ssds must contain u and $k-1$ of its neighbors.*

The existence of a graph for a given strong secure domination number is the main point of the following result.

Theorem 1. *For any two positive integers k and n for $k \leq n$, there exists a graph G of order n with $\gamma_{ss}(G) = k$.*

Proof. Let us construct a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $\gamma_{ss}(G) = k$. When $k = 1$ or $n-1$ or n, choose $G \cong K_n$ or $K_{1,n-1}$ or $\overline{K_n}$, respectively. Let $1 < k \leq \lfloor \frac{n}{2} \rfloor$. Consider the path $P_k = (v_1, v_2, \ldots, v_k)$. Now, partition the remaining $n-k$ vertices into k non-empty sets $V_1, V_2, \ldots, V_k$. Now, construct a graph G from the path P_k in such a way that the subgraph induced by $V_i \cup \{v_i\}$ is complete for all $1 \leq i \leq k$. Then, $\{v_1, v_2, \ldots, v_k\}$ is a γ_{ss}-set of G, and so, $\gamma_{ss}(G) = k$. Let $\lfloor \frac{n}{2} \rfloor < k < n-1$. Consider a path P_2 with its vertices as v_1 and v_2. Partition the remaining $n-2$ vertices into two sets $V_1 = \{v_3, v_4, \ldots, v_{k+1}\}$ and $V_2 = \{v_{k+2}, v_{k+3}, \ldots, v_n\}$. Construct G from the path P_2 in such a way that each vertex of V_1 is adjacent to v_1 and the subgraph induced by $V_2 \cup \{v_2\}$ is complete. Then, $\{v_1, v_2, \ldots, v_k\}$ is a γ_{ss}-set of G. □

3. Trees with Secure Domination Number at Most 4

In this section, we give a necessary and sufficient condition for a tree to have a secure domination number less than or equal to 4. A survey on the constructive characterization of trees using different types of domination numbers was recently provided by the authors in [13].

Let us first review the results for the trees whose strong secure dominance number is either 2 or 3.

Proposition 3 ([11]). *Let T be a tree. Then:*

(i). $\gamma_{ss}(T) = 1$ *if and only if* $T \cong P_2$.
(ii). $\gamma_{ss}(T) = 2$ *if and only if T is either* P_3 *or* P_4.

Let $K_{1,k}$ and $K_{1,l}$ be two stars. A path of length t connects the centers of $K_{1,k}$ and $K_{1,l}$. The resulting graph is represented by $K^t_{k,l}$.

The following theorem characterize all trees whose strong secure dominance number is 3. In what follows, the graphs' $\mathcal{G}_i$s are given in Figure 1.

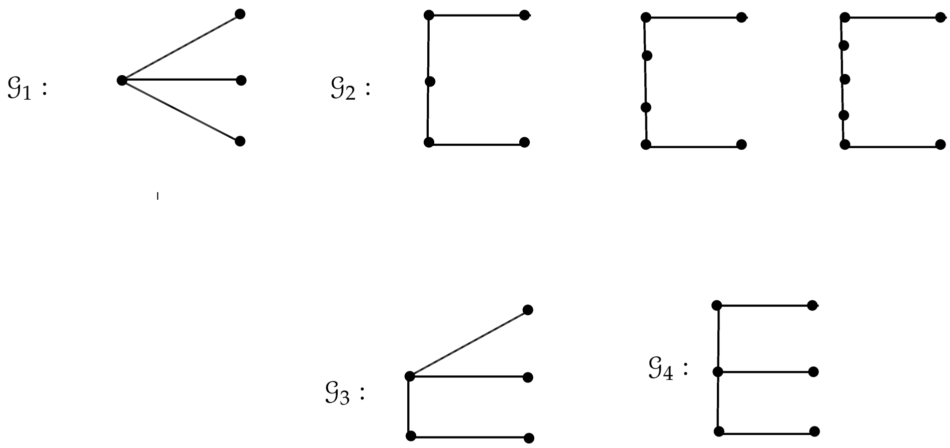

Figure 1. Trees with $\gamma_{ss}(T) = 3$.

Theorem 2. *Let T be a tree. Then, $\gamma_{ss}(T) = 3$ if and only if $T \in \bigcup_{i=1}^{4} \mathcal{G}_i$.*

Proof. Assume that $\gamma_{ss}(T) = 3$. Then, by Proposition 2, T can have at most three support vertices. Suppose T has exactly one support vertex. Then, the support vertex must have 3 pendant neighbors and so $T \cong K_{1,3}$. Assume that T has exactly two support vertices. Then, T is $K_{k,l}^{t}$, where $k, l \geq 1$ and $t \geq 1$. By Proposition 2, we have $k + l \in \{2, 3\}$. If $k + l = 2$, then T is a path. Since $\gamma_{ss}(P_n) = \lceil \frac{3n}{7} \rceil = 3$, we obtain $n \in \{5, 6, 7\}$; these graphs are listed in $\mathcal{G}_2$. If $k + l = 3$, without loss of generality, assume that $k = 2$ and $l = 1$, then $T \cong \mathcal{G}_3$. When T has exactly three support vertices, T is isomorphic to the graph $\mathcal{G}_4$. The converse is obvious. □

The following theorem characterize all trees whose strong secure dominance number is 4. In what follows, the graphs' $\mathcal{F}_i$s are given in Figures 2 and 3.

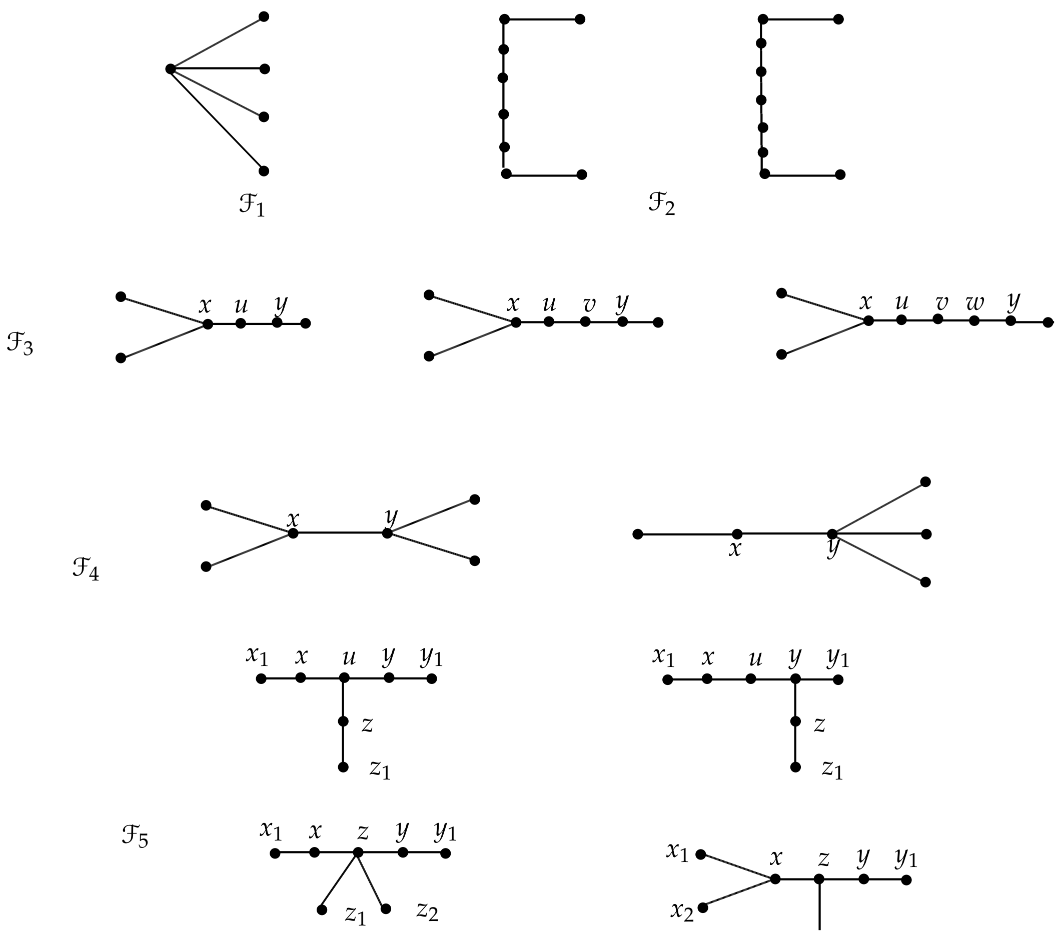

Figure 2. Trees with $\gamma_{ss}(T) = 4$.

Theorem 3. *Let T be a tree. Then, $\gamma_{ss}(T) = 4$ if and only if $T \in \bigcup_{i=1}^{8} \mathcal{F}_i$.*

Proof. Assume that $\gamma_{ss}(T) = 4$. Let X be a minimum strong secure dominating set of T. Then, by Proposition 2, T has at most four support vertices. Suppose T has exactly one support vertex. Then, T is a star graph. Since $\gamma_{ss}(K_{1,n}) = n$, we have $T \cong K_{1,4} = \mathcal{F}_1$. Suppose T has exactly two support vertices. Obviously T becomes $K_{k,l}^{t}$, where $k, l \geq 1$ and $t \geq 1$. By Proposition 2, we have $2 \leq k + l \leq 4$. If $k + l = 2$, then T is a path. In this case, T is isomorphic to a graph in $\mathcal{F}_2$. Let $k + l = 3$. Without loss of generality, let $k = 2$ and $l = 1$. Let x and y be the two support vertices of T. Since $|X| \geq 4$, x and y are connected by a path of length at most 4. In this case, T is isomorphic to a graph in $\mathcal{F}_3$. If $k + l = 4$, then $k = 2$ and $l = 2$ or $k = 1$ and $l = 3$. In this case, T is isomorphic to a tree in $\mathcal{F}_4$. Next, we assume